Complex Analysis

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1 Introduction

It should be very familiar to the reader that the set or Real numbers ($\mathbb{R}$), is closed under addition, subtraction, multiplication and division and that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$, where $\mathbb{N}$ are the natural, $\mathbb{Z}$ the integers and $\mathbb{Q}$ the rational numbers. But, the class of numbers is not yet complete because if we take $\sqrt{-9} \notin \mathbb{R}$, thus we need the class of imaginary numbers, i.e. the real numbers with the imaginary numbers $ai = \sqrt{-1}$ where $a \in \mathbb{R}$ define the set of complex numbers

Definition 1 *A complex number is an expression of the form $a + bi$, where $a, b \in \mathbb{R}$. Two complex numbers $a + bi$ and $c + di$ are said to be equal if and only if $a = c$ and $b = d$.*

The operations of addition and subtraction of complex numbers are given by

$$(a + bi) \pm (c + di) := (a \pm c) + (b \pm d)i$$

The multiplication of two complex numbers is defined by:

$$(a + bi) \cdot (c + di) = (ac - bd) + (bc + ad)i$$

The division of complex numbers is given by:

$$\frac{a+bi}{c+di} = \frac{a+bi}{c+di}\frac{c-di}{c-di} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i \quad (\text{ if } c^2 + d^2 \neq 0)$$

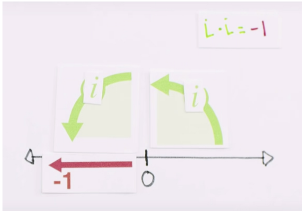
Definition 2 *The real part of the complex number $z = a + bi$ is the real number a and its denoted by $\mathrm{Re}(z)$. Its imaginary part is the real number b, and its denoted by $\mathrm{Im}(z)$. If a is zero, the number is said to be a pure imaginary number.*

1.1 Point representation of complex numbers

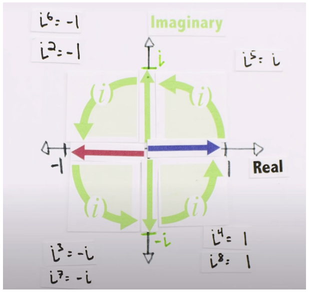
First note the following:

$$\begin{gathered} i^1 = i,\ i^2 = -1,\ i^3 = -i \\ i^4 = i,\ i^5 = -1,\ i^6 = -i \\ \vdots \end{gathered}$$

In the real line is not possible to land after we square in the negative numbers. Thus, we need something in the middle, something that when we multiply by it we rotated by 90 degrees instead of 180 degree. This is, indeed what imaginary numbers do:



The Cartesian coordinate system suggests a convenient way to represent complex numbers ($z = a + bi$) as an ordered pair (a, b), that is, a point on the plane with x-coordinate a and y-coordinate b. In other words, the y-axis represents the imaginary line, and the x-axis the real line.



The xy-plane used for the purpose to describe complex numbers it is referred to as complex plane or z-plane.

By the Pythagorean theorem, the distance form the point $z = a + bi$ to the origin is given by $\sqrt{a^2 + b^2}$

Definition 3 *The absolute value or modulus of the number $z = a + bi$ is denoted by $|z|$ and is given by*

$$|z| = \sqrt{a^2 + b^2}$$

The reader should note that $|z|$ is always a nonnegative real number and that the only complex number whose modulus is zero is the number 0. In addition, given two points z and w $|z \cdot w| = |z| \cdot |w|$

In addition, let $z, w \in \mathbb{C}$ then $|z - w|$ is the distance between the points z and w.

Definition 4 *The complex conjugate of the number $z = a + bi$ is denoted by $\overline{z}$ and is given by*

$$\overline{z} = a - bi$$

Note that $\overline{z}$ is the reflection in the real axis of the point z.

Some properties of complex conjugates:

1. $z + \overline{z} = 2a = 2\,\mathrm{Re}(z)$
2. $z - \overline{z} = 2bi = 2i\,\mathrm{Im}(z)$
3. $\overline{\overline{z}} = z$

4. $\overline{z+w} = \overline{z} + \overline{w}$
5. $\overline{zw} = \overline{z}\,\overline{w}$
6. $\overline{z^n} = \overline{z}^n$
7. $z\overline{z} = a^2 + b^2$ thus $|z|^2 = z\overline{z}$

Note that from (1) and (2) follows that

$$\mathrm{Re}(z) = \frac{z+\overline{z}}{2} \quad \text{and} \quad \mathrm{Im}(z) = \frac{z-\overline{z}}{2i}$$

and that z is real if and only if $\overline{z} = z$ and z is imaginary if and only if $\overline{z} = -z$

1.2 Vectors and Polar form

1.3 Vectors

With each point z in the complex plane we can associate a vector, namely, the directed line segment from the origin to the point z.

Proposition 1 (Triangle inequality) *For any two complex numbers z and w we have*

$$|z+w| \leq |z| + |w|$$

The laws that apply to the vectors in the real plane works accordingly in the complex plane.

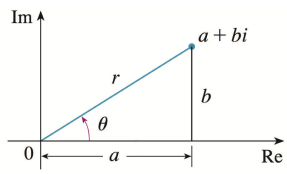
1.3.1 Polar form

Given a complex number $z = a + bi$, we write r for its distance from 0, so

$$r = |z| = \sqrt{a^2+b^2} = \sqrt{z\overline{z}}$$

and θ is the angle of inclination of the vector z, measured positively in counterclockwise sense from the positive real axis. Then we have

$$a = r\cos\theta \quad b = r\sin\theta$$



To find θ we can use that

$$\tan\theta = \frac{b}{a}$$

Note that we cannot always write $\theta = \arctan(b/a)$ since arctan is defined to take values in $(-\pi/2, \pi/2)$. Alternatively, we can solve

$$\cos\theta = \frac{a}{\sqrt{a^2+b^2}} \quad \sin\theta = \frac{b}{\sqrt{a^2+b^2}}$$

The angle θ is called the argument of z and we write $\theta = \arg(z) = \{\theta_0 + 2k\pi | k \in \mathbb{Z}\}$.
We define the principal value of the argument of z as the argument θ in $(-\pi, \pi]$. The principal value of the argument is uniquely defined. It is denoted by $\mathrm{Arg}(z)$.

The choice of the interval $(-\pi, \pi]$ is not unique. In general we define

$$\arg_\tau(z)$$

for the value of the argument in the interval $(\tau, \tau + 2\pi]$. Then

$$\mathrm{Arg}(z) = \arg_{-\pi}(z)$$

with this we can write $z = x + yi$ in the polar form

$$z = r(\cos\theta + i\sin\theta) = r\,\mathrm{cis}\,\theta$$

The polar form is useful to determine the product of two complex numbers z, w:

$$zw = (r_z\,\mathrm{cis}\,\theta_z)(r_w\,\mathrm{cis}\,\theta_w) = (r_z r_w)\,\mathrm{cis}(\theta_z + \theta_w)$$

In other words,

- The modulus of the product is the product of the moduli
- The argument of the product is the sum of the arguments

Which implies that the division is

- The modulus of the division is the division of the moduli
- The arguments of the division is the subtraction of the arguments.

1.4 The complex Exponential

Proposition 2 (Euler's formula) *Let $\theta \in [0, 2\pi]$ then,*

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Thus, let $z \in \mathbb{C}$ in polar form, then

$$z = r\,\mathrm{cis}\,\theta = re^{i\theta} = |z|e^{i\arg(z)}$$

Useful value of the Euler's formula:

$$e^{i0} = 1 \quad e^{i\frac{\pi}{2}} = i \quad e^{i\pi} = -1 \quad e^{i\frac{3\pi}{2}} = -i \quad e^{i2\pi} = 1$$

Definition 5 *If $z = x + iy$, then e^z is defined to be the complex number*

$$e^z = e^x(\cos y + i\sin y)$$

The Euler formula yields to the following representations of the trigonometric functions:

$$\cos\theta = \operatorname{Re} e^{i\theta} = \frac{e^{i\theta} + e^{-i\theta}}{2}$$
$$\sin\theta = \operatorname{Im} e^{i\theta} = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

The rules derived in Sec 1.3 for multiply and divided in polar form find very natural expressions:

$$zw = (r_z r_w)e^{i(\theta_z+\theta_w)} \quad \frac{z}{w} = \left(\frac{r_z}{r_w}\right) e^{i(\theta_z-\theta_w)}$$

and the complex conjugation of $z = re^{i\theta}$:

$$\overline{z} = re^{-i\theta}$$

Proposition 3 (De Moivre's theorem) *Let $z \in \mathbb{C}$,*

$$z^n = r^n(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta) \quad \forall n \in \mathbb{N}$$

1.5 Roots

A complex number w is a an n-th root of the complex number z if

$$w^n = z$$

Proposition 4 *The n-root if the complex number z are given by:*

$$w_k = r^{1/n} e^{i\frac{\theta+2k\pi}{n}}$$

where $k \in \{0, 1, ..., n-1\}$

Remarks:

- For all k we have $|w_k| = r^{1/n}$, i.e all roots lies on the same circle.
- $w_{k+1} = w_k e^{i\frac{2\pi}{n}}$, i.e. each root can be obtain by rotating the previous root by the angle $\frac{2\pi}{n}$
- The roots are equally spaced on the circle of radius $r^{1/n}$, the angle between two successive roots is $2\pi/n$ and the first root has angle θ/n.

1.6 Planar Sets

The first things to notice is that the modulus, i.e $d(z, w) = |z - w|$ where $z, w \in \mathbb{C}$ is a metric (its satisfies the 3 axioms). Therefore, the notions that we have learned in metric and topological space apply here.

Theorem 1 *Suppose $u(x, y)$ is a real-valued function defined in a domain D. If the first partial derivatives of u satisfy*

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$$

at all points of D, then $u \equiv$ constant in D.

1.7 The Riemann Sphere and Stereographic Projection

Complex numbers can be visualized as points on the unit sphere in 3-space (The Riemann sphere) via stereographic projection, which associates with a point z in equatorial plane the point at which the line through z and the north pole cuts the sphere. The extended complex number ∞ is identified with the north pole, and $\mathbb{C} \cup \{\infty\}$ is called the extended complex plane and as proved in metric and topological spaces is compact.

2 Analytic(Holomorphic) Function

Definition 6 *Let $f : U \to \mathbb{C}$ be a complex valued function defined in a neighbourhood of z_0. Then f is differentiable at z_0 if*

$$f'(z_0) = \lim_{\delta z \to 0} \frac{f(z_0 + \delta z) - f(z_0)}{\delta z}$$

exists.

Definition 7 *Let $f : U \to \mathbb{C}$ be a complex valued function defined in a neighbourhood of z_0. Then if f is differentiable at z_0 then*

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + R(z)(z - z_0)$$

where $o(z - z_0) = R(z)$, i.e. R is sublinear.

Theorem 2 *If f is differentiable at z_0 then f is continuous at z_0*

Remark: The differentiability rules (theorems, definition, etc.) in real analysis are the same for the complex analysis. For this reason we omit some definition and theorems.

Definition 8 *Let $U \subset \mathbb{C}$ be open, and $f : U \to \mathbb{C}$. Then,*

- *f is* analytic *(Holomorphic) in U if f is differentiable in all $z \in U$*
- *f is* analytic *(holomorphic) in $z_0 \in U$ if there is a $\delta > 0$ such that f is differentiable in all $z \in D_\delta(z_0)$*
- *f is* entire *if f is analytic in $U = \mathbb{C}$.*
- *$z_0 \in U$ is a* singularity *of f if is not analytic in z_0 but for every $\epsilon > 0$ f is analytic in some $z \in D_\epsilon(z_0)$*

Theorem 3 *For every function $f : U \to \mathbb{C}$ there are unique $u, v : V \to \mathbb{R}$ s.t.*

$$f(x + iy) = u(x, y) + iv(x, y)$$

Theorem 4 (Cauchy-Riemann equations) *Let $U \subset \mathbb{C}$ be open, and $f : U \to \mathbb{C}$. If f is differentiable in $z_0 = x_0 + iy_0 \in U$, then the first partial derivatives of u nad v in (x_0, y_0) exist and we have*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Theorem 5 *Let $U \subset \mathbb{C}$ be open, and $f : U \to \mathbb{C}$. If the first partial differentials of u and v exist on $V = \{(x, y) \in \mathbb{R}^2 \,|\, z = x + iy \in U\}$ and are continuous in $(x_0, y_0) \in V$ and if the Cauchy-Riemann equations at (x_0, y_0) hold. Then, f is analytic in z_0.*

Theorem 6 *Let $U \subset \mathbb{C}$ be open, $f : U \to \mathbb{C}$, and $V = \{(x, y) \in \mathbb{R}^2 \,|\, z = x + iy \in U\}$. Then f is analytic in U if and only if u and v are continuously differentiable in V and if*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

as functions in V.

Theorem 7 *If $f(z)$ is analytic in a domain D and if $f'(z) = 0$ everywhere in D, then f is constant in D*

3 Harmonic function

Definition 9 *A harmonic function on $V \subset \mathbb{R}^2$ is a twice continuously differentiable function $u : V \to \mathbb{R}$ such that the Laplace's equation holds*

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

Theorem 8 *Let $f : U \subset \mathbb{C} \to \mathbb{C}$ be analytic, where $f = u + iv$. Then u, v are harmonic functions*

Theorem 9 *Let $B_1(0) = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\}$, $u : B_1(0) \to \mathbb{R}$ be harmonic. Then there exists a function $v : B_1(0) \to \mathbb{R}$ such that $f = u(x, y) + iv(x, y) : B_1(0) \to \mathbb{C}$ is analytic. Moreover, the function v is harmonic and unique up to adding a constant.*

Remark: Such v is called a harmonic conjugate of u.

4 The Logarithmic Function, complex powers, inverses

Definition 10 (Principal Branch) *For every $\mathbb{C} \backslash \mathbb{R}_{\leq 0}$ there is a unique $y \in (-\pi, \pi)$ such that $w = |w| \cdot e^{iy} = e^{x+iy}$ where $x = ln|w|$*

$$log : \mathbb{C} \backslash \mathbb{R}_{\leq 0} \to \mathbb{C}$$
$$w \mapsto log(w) = x + iy$$

in other words, $log(w) = ln|w| + iarg\,w$

Definition 11 (General Branch) *The concept is the same as the principal branch but,*

$$Log(w) = ln|w| + iarg\,w + i2k\pi$$

where $k = 0, \pm 1, \pm 2, ...$

Theorem 10 *The function* $Log\, z$ *is analytic in* $\mathbb{C}\backslash\mathbb{R}_{\leq 0}$ *and*

$$\frac{d}{dz}Log\, z = \frac{1}{z}$$

for every $z \in \mathbb{C}\backslash\mathbb{R}_{\leq 0}$

Definition 12 *If* α *is a complex constant and* $z \neq 0$*, then*

$$z^{\alpha} = e^{\alpha log\, z}$$

Definition 13 *The inverses of the cosine, tangent and sine are given by (in their respectively principals branches)*

$$\begin{aligned}\arccos : \mathbb{C}\backslash\{x \in \mathbb{R} | |x| \geq 1\} &\to \mathbb{C}\\ z &\to -ilog(z + i\sqrt{i - z^2})\end{aligned}$$

$$\begin{aligned}\arcsin : \mathbb{C}\backslash\{x \in \mathbb{R} | |x| \geq 1\} &\to \mathbb{C}\\ z &\to -ilog(iz + \sqrt{i - z^2})\end{aligned}$$

$$\begin{aligned}\arctan : \mathbb{C}\backslash\{iy \in \mathbb{R} | |y| \geq 1\} &\to \mathbb{C}\\ z &\to -\frac{i}{2}log\left(\frac{i - z}{i + z}\right)\end{aligned}$$

5 Polynomial, Power series, Taylor series

Definition 14 *A complex polynomial function is a function of the form*

$$\begin{aligned}f : \mathbb{C} &\to \mathbb{C}\\ z &\mapsto c_n z^n + \cdots + c_1 z + c_0\end{aligned}$$

It's degree is $degf = \max\{i \in \{0, ..., n\} | c_i \neq 0\}$*. A root of* f *is an* $z \in \mathbb{C}$ *s.t.* $f(z) = 0$

Theorem 11 (Fundamental Theorem of Algebra) *Every nonconstant polynomial with complex coefficients has at least one zero.*

Lemma 15 $f : \mathbb{C} \to \mathbb{C}$ *polynomial. Then* f *is entire.*

Definition 16 *A complex rational function is a function of the form*

$$\begin{aligned}\frac{f}{g} : \mathbb{C}\backslash\{z_1, ..., z_d\} &\to \mathbb{C}\\ z &\mapsto \frac{f(z)}{g(z)}\end{aligned}$$

where f *and* g *are polynomial, and* $\{z_1, ..., z_d\}$ *are roots of* g *which are also called the poles of* f/g*.*

Lemma 17 *f, g are polynomial. Then f/g is analytic*

Definition 18 *A complex power series is an expression of the form $\sum_{n=0}^{\infty} a_n z^n$ where $a_i \in \mathbb{C}$ for all $i \in \mathbb{N}$.*

Definition 19 *The $N-th$ partial sum of $\sum_{n=0}^{\infty} a_n z^n$ is $S_N = \sum_{n=0}^{N} a_n z^n$*

Remark: Note that $a_n z^n = S_n - S_{n-1}$

Definition 20 *A complex power series converges to $w \in \mathbb{C}$ if $w = \lim_{N\to 0} S_N$*

Definition 21 *A power series converges absolutely if $\sum_{n=0}^{\infty} |a_n||z^n|$ converges in $\mathbb{R}$.*

Definition 22 *A complex power series converges absolutely in $D_\delta(0)$ if it converges absolutely for all $z \in D_\delta(0)$*

Theorem 12 *If a complex power series converges absolutely, then it converges in $\mathbb{C}$.*

Remark: Most of theorems and definitions from real analysis still holds for the complex case.

Theorem 13 *The geometric series $\sum_{n=0}^{\infty} z^n$ converges absolutely in $D_1(0)$ and*

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$

for $z \in D_1(0)$.

Theorem 14 (Cauchy-Hadamard) *Let a power series $\sum_{n=0}^{\infty} a_n z^n$, then the radius of convergence is given by*

$$R = \frac{1}{\limsup |a_n|^{\frac{1}{n}}}$$

Moreover, $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely if $|z| < R$ (diverges otherwise).

Remark: For $|z| = R$, the power series could diverges or converges.

Theorem 15 (Comparison Test) *Suppose that the terms c_j satisfies the inequality*

$$|c_j| \leq M_j$$

for all integers j larger than some number J. Then if the series $\sum_{j=0}^{\infty} M_j$ converges, so does $\sum_{j=0}^{\infty} c_j$

Theorem 16 *(Ratio test) Suppose that the terms of the series $\sum_{j=0}^{\infty} c_j$ have the property that the ratios*

$$\left| \frac{c_{j+1}}{c_j} \right|$$

approach a limit L as $j \to \infty$ Then the series converges if $L < 1$ and divers otherwise $(L > 1)$

Theorem 17 *Given two absolutely convergent power series, then the sum and the product absolutely converges. Moreover, if a power series absolutely converges, then the product with a constant absolutely converges.*

Theorem 18 $f(z) = \sum_{n=0}^{\infty} a_n z^n : D_R(0) \to \mathbb{C}$ *is absolutely convergent. Then* f *is analytic on* $D_R(0)$ *and*

$$f'(z) = \sum (n+1) a_{n+1} z^n$$

In particular the RHS converges absolutely on $D_R(0)$ *and has the same radius of convergent of* f

Definition 23 (Power series around a point) *A power series around* $z_0 \in \mathbb{C}$ *is an expression of the form*

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

with $a_n \in \mathbb{C}$ *for all* $n \geq 0$

Theorem 19 (Cauchy-Hadamard) *A power series around* $z_0 \in \mathbb{C}$ *have the radius of convergent given by* $R = (\limsup |a_n|^{1/n})^{-1}$. *Then, the power series converges absolutely on* $D_R(z_0)$ *and defines an analytic function* $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n : D_R(z_0) \to \mathbb{C}$ *with derivative* $f'(z) = \sum_{n=0}^{\infty} (n+1) a_{n+1} (z - z_0)^n$

Corollary 24 $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n : D_R(z_0) \to \mathbb{C}$ *is arbitrary often differentiable and*

$$a_n = \frac{1}{n!} f^{(n)}(z_0)$$

Definition 25 (Taylor series development) *Let* $f : U \subset \mathbb{C} \to \mathbb{C}$ *arbitrarily often differentiable. Then the Taylor series development of* f *around* $z_0 \in \mathbb{C}$ *is the power series*

$$\sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_0)(z - z_0)^n$$

Theorem 20 (Taylor development of analytic functions) *Let* $f : U \subset \mathbb{C} \to \mathbb{C}$ *analytic and* $\overline{D}_R(z_0) \subset U$.

1. *Then* f *is arbitrary often differentiable and for all* $z \in D_R(z_0)$

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_0)(z - z_0)^n$$

in particular, the Taylor series of f *converges on* $D_R(z_0)$

2. *Let* $\gamma = \partial \overline{D}_R(z_0) = \{z_0 + Re^{it} \mid t \in [0, 2\pi]\}$. *Then*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(z)}{(z - z_0)^{n+1}} dz$$

for all $n \in \mathbb{N}$ *and* $z \in \overline{D}_R(z_0)$

Theorem 21 *$f : D_R(z_0) \to \mathbb{C}$ analytic. Then f is arbitrary often differentiable and*

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_0)(z - z_0)^n$$

for all $z \in D_R(z_0)$

Theorem 22 *$f : D_R(z_0) \subset U \subset \mathbb{C} \to \mathbb{C}$ analytic. Then the radius of converges of the Taylor development is $\geq R$.*

Corollary 26 *$f : U \subset \mathbb{C} \to \mathbb{C}$ analytic. Then f is arbitrary often differentiable on U.*

Definition 27 (Bounded) *A function $f : U \to \mathbb{C}$ is bounded if there is a bound $M \in R$ such that $|f(z)| \leq M$ for all $z \in U$*

Theorem 23 *Let f be analytic inside and on a circle C_r or radius r centered about z_0. If $|f(z)| \leq M$ for all z on C_r, then the derivatives of f at z_0 satisfy*

$$\left| f^{(n)}(z_0) \right| \leq \frac{n!M}{r^n}$$

Theorem 24 (Liouville's theorem) *The only bounded entire functions are the constant functions*

Lemma 28 *Suppose $f : D_r(z_0) \to \mathbb{C}$ analytic, if $|f(z)| \leq |f(z_0)|$, then f is constant.*

Theorem 25 *A function analytic in a bounded domain and continuous up to and including its boundary points attains its maximum modulus on the boundary*

Theorem 26 (Maximums modulus principles) *$K \subset \mathbb{C}$ compact, and $f : K \to \mathbb{C}$ continuous and analytic on $U = Int(K)$ interior of K. Then, $|f| : K \to \mathbb{C}$, $z \mapsto |f(z)|$ assumes its maximum on ∂K.*

Proposition 5 (Polynomial division) *Let $g = \sum_{n=0}^{deg\, g} a_n z^n$ and $h = \sum_{n=0}^{deg\, h} b_n z^n$ complex polynomials, $h \neq 0$. Then, there are complex polynomials q and r such that $g = q \cdot h + r$ and $deg\, r < deg\, h$ or $r = 0$.*

Corollary 29 *$g = \sum_{n=0}^{d} a_n z^n$ complex polynomial of degree d. Then $g = a_d \prod_{i=1}^{d}(z - z_i)$ for some $z_i, ..., z_d \in \mathbb{C}$.*

Remark: if $g = a_d \prod_{i=1}^{d}(z - z_i)$, then $g(z) = 0$ iff $z \in \{z_1, ..., z_d\}$

Theorem 27 (Cauchy Product) *Let $\sum_{i=0}^{\infty} a_i$ and $\sum_{j=0}^{\infty} b_j$ be two infinite series with complex terms. Then, the Cauchy product is defined as*

$$\left(\sum_{i=0}^{\infty} a_i \right) \cdot \left(\sum_{j=0}^{\infty} b_j \right) = \sum_{k=0}^{\infty} \sum_{l=0}^{k} a_l b_{k-l}$$

Theorem 28 (Partial fractional decomposition) *g, h complex polynomials, $h = c \cdot \prod_{i=1}^{s}(z - z_i)^{n_i} \neq 0$ with $z_1, ..., z_s$ pairwise distinct and $n_i, ..., n_s \geq 1$. Then there are a polynomial q and $c_{i,-n_i}, ..., c_{i,-1} \in \mathbb{C}$ for $i = 1, ..., s$ such that*

$$\frac{g}{h} = q + \sum_{i=1}^{s} \sum_{k=-n_i}^{-1} c_{i,k}(z - z_i)^k$$

Remark: Let $g/h = \sum_{k=-n_i}^{\infty} c_{i,k}(z - z_i)^k$ on $D_\epsilon^\circ(z_i)$ where

$$c_{i,k} = \frac{1}{(n_i - k)!} \cdot \frac{d^{n_i - k}}{dz^{n_i - k}} \left[(z - z_i)^{n_i} \frac{g}{h}\right]_{z = z_i}$$

6 Complex Integration

Definition 30 *A parametrized arc is a map $z : [a, b] \to \mathbb{C}$ $t \mapsto z(t)$. Then $z(t) = x(t) + iy(t)$ for*

$$\begin{aligned} x : [a,b] &\to \mathbb{R} \qquad \textit{and} \qquad & y : [a,b] &\to \mathbb{R} \\ t &\mapsto Re(z(t)) & t &\mapsto Im(z(t)) \end{aligned}$$

Definition 31 *z is continuously differentiable if x and y are continuously differentiable.*

Definition 32 (Smooth arc (or curve)) *A smooth arc is the image $\gamma = \{z(t) | t \in [a, b]\}$ of a parametrization $z : [a, b] \to \mathbb{C}$ $a < b$ that satisfies*

(1) It is continuously differentiable; (no corners and jumps)

(2) $z'(t) \neq 0$ for all $t \in [0, b]$; (no stopping)

(3) it is injective (no selfintersection)

together with the order $\gamma(s) < \gamma(t)$ for $a \leq s \leq t \leq b$ (orientation).

Lemma 33 *$z : [a, b] \to \mathbb{C}$, $\tilde{z}[\tilde{a}, \tilde{b}] \to \mathbb{C}$ parametrizations of the same smooth arc γ, i.e. $z([a, b]) = \gamma = \tilde{z}([\tilde{a}, \tilde{b}])$, $z(a) = \tilde{a}$, $z(b) = \tilde{b}$ and $z, \tilde{z}$ satisfy (1)-(3).*
Then there exists a unique continuously differentible bijection $\sigma : [a, b] \to [\tilde{a}, \tilde{b}]$ such that $z(t) = \tilde{z}(\sigma(t))$ for all $t \in [a, b]$.

Definition 34 (Length) *Let γ be a smooth arc with parametrization $z : [a, b] \to \mathbb{C}$. The length of γ is*

$$l(\gamma) = \int_a^b |z'(t)| dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2} dt$$

Proposition 6 *If γ is a smooth arc then $l(\gamma)$ is independent from the choice of parametrization $z : [a, b] \to \mathbb{C}$*

Definition 35 *$U \subset \mathbb{C}$ open, $f : U \to \mathbb{C}$ continuous, $f = u + iv$ and $z : [a, b] \to \mathbb{C}$ smooth arc with $\gamma = \mathrm{Im}(z) \subset U$, $z = x + iy$. Then the integral of f along γ is*

$$\int_\gamma f = \int_\gamma f(z)dz = \int_a^b f(z(t))z'(t)dt$$

Remark: The integral is equal to

$$\int_\gamma f = \int_a^b v(x(t), y(t))x'(t) + i^2 v(x(t), y(t))y'(t)\, dt + i \int_a^b v(x(t), y(t))y'(t) + v(x(t), y(t))x'(t)\, dt$$

Proposition 7 *γ smooth arc then $\int_\gamma f(z)dz = \int_a^b f(z(t))z'(t)dt$ is independent from the choice of parametrization $z : [a, b] \to \mathbb{C}$*

Definition 36 (Contour) *A contour is the image Γ of a continuous map $z : [a, b] \to \mathbb{C}$ $(a \leq b)$ for which there is a partition $a = \alpha_0 < \cdots < \alpha_n = b$ such that for every $i = 1 - n$ $\gamma_i = \{z(t)|t \in [a_{i-1}, a_i]\}$ is a smooth arc, together with orientation $z(s) < z(t)$ for $a \leq s < t \leq b$ we write $\Gamma = \gamma_i + \cdots + \gamma_n$*

Definition 37 *A countour Γ with parametrization $z : [a, b] \to \mathbb{C}$ is closed if $z(a) = z(b)$*

Definition 38 *$U \subset \mathbb{C}$ open, $f : U \to \mathbb{C}$ continuous, $\Gamma = \gamma_i + \cdots + \gamma_n \subset U$ contour. Then*

$$\int_\Gamma f = \int_\gamma f(z)dz = \sum_{i=1}^n \int_{\gamma_i} f(z)dz$$

Proposition 8 *$\int_\Gamma f(z)dz$ does not depend on the choice of $\Gamma = \gamma_i + \cdots + \gamma_n$*

Definition 39 (Primitive) *$U \subset \mathbb{C}$ open, $f : U \to \mathbb{C}$. A primitive of f is an analytic function $F : U \to \mathbb{C}$ such that $f = F'$*

Theorem 29 *$f : U \to \mathbb{C}$ continuous with primitive F. $\Gamma \subset U$ contour with parametrization $z : [a, b] \to U$ then*

$$\int_\Gamma f = F(z(b)) - F(z(a))$$

Proposition 9 *$U \subset \mathbb{C}$ open, $f, g : U \to \mathbb{C}$ continuous functions. Γ contour with parametrization $z : [a, b] \to U$ $-\Gamma$ inverse contour, given by $z^- : [a, b] \to U$ $t \mapsto a + b - t$. Then*

- $\int_\Gamma (f + g) = \int_\Gamma f + \int_\Gamma g$
- $\int_\Gamma cf = c \int_\Gamma f$
- $\int_{-\Gamma} f = -\int_\Gamma f$

Proposition 10 *$U \subset \mathbb{C}$ open, $f : U \to \mathbb{C}$ continuous, $\Gamma \subset U$ contour of legth $l(\Gamma)$ and $M = \max\{|f(z)| \,|\, z \in \Gamma\}$ then*

$$\left| \int_\Gamma f \right| \leq Ml(\gamma)$$

Remark:Γ is compact and $|f| : \Gamma \to \mathbb{R}$ $z \mapsto f(z)$ is continuous. Thus there is a $z_i \in \Gamma$ such that $|f(z)| \leq f(z_1) = M$ for all $z \in \Gamma$

Theorem 30 *$f : D_r(z_0) \to \mathbb{C}$ analytic and Γ_z Line from z_0 to $z \in D_r(z_0)$ then*

$$F : D_r(z_0) \to \mathbb{C} \quad z \mapsto \int_{\Gamma_z} f$$

is a primitive of f

Definition 40 (Homotopy) *Let a open subset U of $\mathbb{C}$, and $z, w : [a, b] \to U$ continuous with $z(0) = w(0)$ and $z(b) = w(b)$. A* Homotopy *between z and w (with fixed end points) is a continuous map*

$$\begin{aligned} H : [0,1] \times [a,b] &\to U \\ (s,t) &\mapsto H(s,t) = H_s(t) \end{aligned}$$

such that

- $H_0(t) = z(t)$ *for all* $t \in [a, b]$
- $H_1(t) = w(t)$ *for all* $t \in [a, b]$
- $H_s(a) = z(a) = w(a)$ *for all* $s \in [0, 1]$
- $H_s(b) = z(b) = w(b)$ *for all* $s \in [0, 1]$

Theorem 31 *z and w are homotopic if there is a homotopy H between z and w*

Theorem 32 *Let f be analytic in a open subset U of $\mathbb{C}$, and $z, w : [a, b] \to U$ continuous with $\gamma_z = z([a, b])$ and $\gamma_w = w([a, b])$. If z and w are homotopic, then*

$$\int_{\gamma_z} f = \int_{\gamma_w} f$$

Theorem 33 (Deformation Invariance Theorem) *Let f be a function analytic in a domain D containing the loops Γ_0 and Γ_1. If these loops can be continuously deformed into one another in D, then*

$$\int_{\Gamma_0} f(z)\, dz = \int_{\Gamma_1} f(z)\, dz$$

Definition 41 *Let $U \subset \mathbb{C}$ be open, and $z : [a, b] \to U$ closed. Then z is contractible (in U) if z is homotopic to the constant path*

$$\begin{aligned} c_{z_0} : [a,b] &\to U \\ t &\mapsto z_0 \end{aligned}$$

Definition 42 (simply connected domain) *Any domain D possessing the property that every loop in D can be continuously deformed in D to a point is called* simply connected domain

Theorem 34 (Cauchy) *Let $U \subset \mathbb{C}$ open, $f : U \to \mathbb{C}$ analytic and $\Gamma \subset U$ closed contour. If Γ is contractible, then*

$$\int_\Gamma f(z)dz = 0$$

Theorem 35 (Cauchy) *If f is analytic in a simply connect domain D and Γ is any loop in D, then*

$$\int_\Gamma f(z)dz = 0$$

Theorem 36 (Goursat) *$U \subset \mathbb{C}$ open and $f : U \to \mathbb{C}$ analytic. T is a solid triangle in U and $\Gamma = \partial T$ (+ counterclockwise orientation). Then*

$$\int_\Gamma f(z)dz = 0$$

Corollary 43 *If f is analytic in a domain D and Γ is a loop then*

$$\int_\Gamma f(z)dz = 0$$

Theorem 37 (Cauchy's Integral Formula) *Let Γ be a simple closed positively oriented contour. If f is analytic in some simply connected domain D containing Γ and z_0 is any inside Γ, then*

$$f(z_0) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - z_0}\, dz$$

Theorem 38 (Generalized Cauchy's integral formula) *Let Γ be a simple closed positively oriented contour. If f is analytic in some simply connected domain D containing Γ and z_0 is any inside Γ, then*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(z)}{(z - z_0)^{n+1}} dz$$

Theorem 39 *If f is analytic in a domain D, then all its derivatives exist and are analytic in D*

Theorem 40 *If $f = u + iv$ is analytic in a domain D, then all partials derivatives exist and are continuous in D*

Theorem 41 *If f is continuous in a domain D and if*

$$\int_\Gamma f(z)\, dz = 0$$

for every closed contour Γ in D, then f is analytic in D

Corollary 44 (Mean value principle) *Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$ analytic in $D_r(z_0) \subset U$. Then*

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})\, dt$$

6.1 Improper integrals

Proposition 11 *Let $U \subset \mathbb{C}$ be open and connected, assume that $f : U \to \mathbb{C}$ is analytic. Take the sequence $\{w_n\}_{n\in\mathbb{N}} \subset U\backslash\{z_0\}$ that converges to $z_0 \in U$. If $f(w_n) = 0$ for all $n \in \mathbb{N}$, then f is the zero function.*

Corollary 45 *$U \subset \mathbb{C}$ domain such that $(a, b) \subset U \cap \mathbb{R}$. Take $f, g : U \to \mathbb{C}$ be analytic. If $f|_{(a,b)} = g|_{(a,b)}$, then $f = g$ on U*

Corollary 46 *Let $f : U \to \mathbb{C}$ where $f \not\equiv 0$. Take $k \subset \mathbb{C}$ compact such that $K\backslash U \subset \{$poles of $f\}$ then f has finitely many zeros and poles in K*

Lemma 47 *Let $f : \mathbb{R} \to \mathbb{C}$ continuous. If there are $c, r > 0$ such that*

$$|f(x)| \leq \frac{c}{|x|^2}$$

for all $x \in \mathbb{R}$ with $|x| \geq r$, then

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{t\to\infty} \int_{-t}^{t} f(x)dx$$

where the $\int_{-\infty}^{\infty} f(x)dx$ converges.

Remark: It can happen that the limit exists, but the improper integral diverges.

Theorem 42 *Let $U \subset \mathbb{C}$ open with $\mathbb{R} \subset U$, and $S = \{z \in \mathbb{C}\backslash U \mid \operatorname{Im} z > 0\}$. Take $f : U \to \mathbb{C}$ analytic such that $S \subset \{$poles of $f\}$. If there are $c, r > 0$ such that $|f(x)| \leq \frac{c}{|x|^2}$ for all $z \in U$ with $\operatorname{Im} z \geq 0$, then*

$$\int_{-\infty}^{\infty} f(x)dx$$

converges, S is finite and

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \cdot \sum_{z\in S} \operatorname{Res}_z f$$

Remark: If $f(\mathbb{R}) \subset \mathbb{R}$, then

$$\sum_{z\in S} \operatorname{Res}_z f = \frac{1}{2\pi i} \int_{-\infty}^{\infty} f(x)dx$$

is in $i\mathbb{R} = \{z \in C \,|\, Re\, z = 0\}$

Lemma 48 *Let g, h polynomials with $deg\, h \geq deg\, g + 2$. Then there are $c, r > 0$ such that*

$$\left|\frac{g(z)}{h(z)}\right| \leq \frac{c}{|z|^2}$$

for all $|z| \geq r$

Theorem 43 *g, h polynomials with $\deg h \geq \deg g + 1$, $h(x) \neq 0$ for $x \in \mathbb{R}$. Then*

$$\int_{-\infty}^{\infty} \frac{g(x)}{h(x)} e^{ikx} dx = 2\pi i \sum_{\substack{h(z_0)=0 \\ k\cdot \mathrm{Im}(z_0)>0}} \mathrm{Res}_{z_0} \left(\frac{g(z)}{h(z)} e^{ikz} \right)$$

Theorem 44 *Let $f(x,y) : U \to \mathbb{R}$ real rational function with $\{\cos\theta, \sin\theta) \,|\, t \in [0, 2\pi]\} \subset U$ Then*

$$\int_0^{2\pi} f(\cos\theta, \sin\theta) d\theta = 2\pi i \cdot \sum_{w \in D_1(0)} \mathrm{Res}_w \left(\frac{1}{iz} f\left(\frac{z^2+1}{2z}, \frac{z^2-1}{2iz} \right) \right)$$

7 Zeros, Poles and Laurent series

Definition 49 (Zero) *Let $f : U \subset \mathbb{C} \to \mathbb{C}$ analytic. A* zero *(or root) of f is a $z_0 \in U$ such that $f(z_0) = 0$. The* order *of a zero z_0 of f is*

$$\mathrm{ord}_{z_0}(f) = \max\{n \in \mathbb{N} \,|\, f^{(i)}(z_0) = 0 \text{ for } i = 0, ..., n-1\}$$

A simple zero *is a zero of order* 1

Remark: $\mathrm{ord}_z(f)$ makes also sense if z is not a zero, and is also called the *order of f* in z

Remark: Since $f(z) = f^{(0)}(z)$, z is a zero of f iff $\mathrm{ord}_z(f) \geq 1$

Definition 50 (Laurent series at z_0) *A Laurent series at z_0 is a expression of the form*

$$\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n = \underbrace{\sum_{n=-\infty}^{-1} a_n (z-z_0)^n}_{\textit{principal part}} + \underbrace{\sum_{n=0}^{\infty} a_n (z-z_0)^n}_{\textit{regular (or analytic) part}}$$

with $a_n \in \mathbb{C}$ for $n \in \mathbb{Z}$

Remark:

$$\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n = \underbrace{\sum_{n=1}^{\infty} a_{-n} (z-z_0)^{-n}}_{r:=(\text{radius of convergence})^{-1}} + \underbrace{\sum_{n=0}^{\infty} a_n (z-z_0)^n}_{R:=(\text{radius of convergence})}$$

converges for all $z \in \mathrm{Ann}_{r,R}(z_0) := \{z \in \mathbb{C} \,|\, r < |z - z_0| < R\}$ (annulus).

Theorem 45 (Laurent development on annuli) *$f : \mathrm{Ann}_{r,R}(z_0) \to \mathbb{C}$ analytic. $\gamma = \{z_0 + ge^{it} \,|\, t \in [0, 2\pi]\}$ for some $g \in (r, R)$ and*

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-z_0)^{n+1}} dz$$

then

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

Remark: The Laurent representation is unique.

Definition 51 (Isolated singularity) *Let $f : U \subset \mathbb{C} \to \mathbb{C}$ analytic. An isolated singularity of f is a point $z_0 \in \mathbb{C} \backslash U$ such that the* punctuated disc

$$D_\epsilon^\circ(z_0) = \mathrm{Ann}_{0,\epsilon} = \{z \in \mathbb{C} \,|\, 0 < |z - z_0| < \epsilon\}$$

is contained in U for some $\epsilon > 0$

Remark: If $D_\epsilon^\circ(z_0) \subset U$, then f has a unique Laurent development

Definition 52 *Let $f : U \subset \mathbb{C} \to \mathbb{C}$ analytic, $D_\epsilon^\circ(z_0) \subset U$ and $f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ for some $z \in D_\epsilon^\circ(z_0)$. The* order of f *in z_0 is*

$$\mathrm{ord}_{z_0}(f) = \inf\{n \in \mathbb{Z} \,|\, a_n \neq 0\}$$

Then z_0 is

- *a* removable singularity *if* $\mathrm{ord}_{z_0}(f) \geq 0$ *(i.e. $a_j = 0$ for all $j < 0$)*
- *a* pole (of order *n) if $-n = \mathrm{ord}_{z_0}(f) < 0$, but $\neq -\infty$ (i.e. $a_{-m} \neq 0$ for some positive integer m but $a_j = 0$ for all $j < -m$)*
- *an* essential singularity *if* $\mathrm{ord}_{z_0}(f) = -\infty$

Theorem 46 *Let f be analytic at z_0. Then f has a zero of order m at z_0 if and only if f can be written as $f(z) = (z - z_0)^m g(z)$, where g is analytic at z_0 and $g(z_0) \neq 0$.*

Proposition 12 *$f, g : D_\epsilon^\circ(z_0) \subset U \to \mathbb{C}$ nonzero analytic. Then,*

- $\mathrm{ord}_{z_0}(f \cdot g) = \mathrm{ord}_{z_0}(f) \cdot \mathrm{ord}_{z_0}(g)$
- $\mathrm{ord}_{z_0}\left(\frac{1}{f}\right) = -\mathrm{ord}_{z_0}(f)$
- $\mathrm{ord}_{z_0}(f + g) \geq \min\{\mathrm{ord}_{z_0}(f), \mathrm{ord}_{z_0}(g)\}$

Theorem 47 (Riemann extension theorem) *$f : U \subset \mathbb{C} \to \mathbb{C}$ analytic. Assume $z_0 \in \mathbb{C} \backslash U$ isolated singularity of f. If*

$$\sup_{z \in D_\epsilon^\circ(z_0)} |f(z)| < \infty$$

for some $\epsilon > 0$, then z_0 is removable singularity *and f extends analytically to z_0*

Proposition 13 *$f : U \subset \mathbb{C} \to \mathbb{C}$ analytic. Assume $z_0 \in \mathbb{C} \backslash U$ pole (of order $-m = \mathrm{ord}_{z_0}(f)$) of f. Then*

$$\lim_{z \to z_0} |f(z)| \to \infty$$

i.e., for every $R > 0$ there is an $\epsilon > 0$ s.t. $f(D_\epsilon^\circ(z_0)) \subset \mathrm{Ann}_{R,\infty}(0)$

Theorem 48 (Casaroti-Weirtstrab) *$f : U \subset \mathbb{C} \to \mathbb{C}$ analytic. Assume $z_0 \in \mathbb{C} \backslash U$ essential singularity of f. Then $f(D_\epsilon^\circ(z_0))$ is dense in $\mathbb{C}$ for every $\epsilon > 0$*

Proposition 14 *$f, g : D_r^\circ(z_0) \to \mathbb{C}$ analytic with Laurent developments $f(z) = \sum_{n=m}^\infty a_n(z-z_0)^n$ and $g(z) = \sum_{n=m}^\infty b_n(z-z_0)^n$ on $D_r^\circ(z_0)$. Then*

$$(f \pm g)(z) = \sum_{n=m}^{\infty}(a_n \pm b_n)(z-z_0)^n \quad (f \cdot g)(z) = \sum_{n=2m}^{\infty}\left(\sum_{\substack{k+l=m \\ k,l\geq m}} a_k \cdot b_l\right)(z-z_0)^n$$

Definition 53 (Residue) *$f : D_\epsilon^\circ(z_0) \subset U \to \mathbb{C}$ analytic with Laurent development $f(z) = \sum_{n=m}^\infty a_n(z-z_0)^n$ on $D_\epsilon^\circ(z_0)$. Then residue of f at z_0 is*

$$\mathrm{Res}_{z_0}(f) = a_{-1}$$

Remark: If f is analytic in z_0, i.e. $z_0 \in U$, then $\mathrm{Res}_{z_0} f = 0$

Remark: If f has a simple pole in z_0, i.e. $\mathrm{ord}_{z_0}(f) = -1$ then

$$f = \frac{\mathrm{Res}_{z_0} f}{z - z_0} + a_0 + a_1(z-z_0) + \cdots$$

on $D_\epsilon^\circ(z_0)$

Proposition 15 *$f : D_\epsilon^\circ(z_0) \subset U \to \mathbb{C}$, and $\gamma = C_r(z_0) \subset D_\epsilon^\circ(z_0)$. Then,*

$$\int_\gamma f = 2\pi i \cdot \mathrm{Res}_{z_0} f$$

Lemma 54 *$f : U \to \mathbb{C}$ analytic and z_0 pole of f of order $m \geq 1$. Then,*

$$\mathrm{Res}_{z_0} f = \frac{1}{(m-1)!} \cdot \frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]_{z=z_0}$$

Lemma 55 *$f : U \to \mathbb{C}$ analytic with a simple pole at z_0. Let $g : U \cup \{z_0\} \to \mathbb{C}$ analytic. Then*

$$\mathrm{Res}_{z_0} fg = g(z_0) \cdot \mathrm{Res}_{z_0} f$$

Definition 56 *$f : U \to \mathbb{C}$ analytic and $S = \{z \in U \mid f(z) = 0\}$ zeros of f. The logarithmic derivative of f is*

$$\frac{f'}{f} : U \backslash S \to \mathbb{C}$$

Proposition 16 *$f : D_\epsilon^\circ \subset U \to \mathbb{C}$ with $\mathrm{ord}_{z_0} f \neq -\infty$ and $f(z) \neq 0$ for $z \in D_\epsilon^\circ(z_0)$. Then*

- $\mathrm{ord}_{z_0} \frac{f'}{f} = -1$ *if* $\mathrm{ord}_{z_0} f \neq 0$
- $\mathrm{ord}_{z_0} \frac{f'}{f} \geq 0$ *if* $\mathrm{ord}_{z_0} f = 0$
- $\mathrm{Res}_{z_0} \frac{f'}{f} = \mathrm{ord}_{z_0}(f)$

in other words,

$$\frac{f'}{f} = \frac{\operatorname{ord}_{z_0} f}{z - z_0} + \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

on $D^\circ_\epsilon(z_0)$ for some $a_0, a_1, ... \in \mathbb{C}$

Definition 57 (winding number) *Let γ a closed arc in $\mathbb{C}$, and assume $z \in \mathbb{C}\backslash\gamma$. The* ***winding number*** *of γ around z_0 is*

$$W(\gamma, z) = \frac{1}{2\pi i} \int_\gamma \frac{1}{w - z} dw$$

Theorem 49 (Residue formula) *Let $U \subset \mathbb{C}$ be open, and $\gamma \subset U$ be a contractible closed arc. Assume $f : U\backslash\{z_1, ..., z_s\} \to \mathbb{C}$ analytic, where $\{z_1, ..., z_s\} \in U\backslash\gamma$ are poles of f. Then,*

$$\int_\gamma f = 2\pi i \sum_{i=1}^{s} W(\gamma, s_i) \cdot \operatorname{Res}_{z_i} f$$

Remark: The above theorems works also if $\{z_1, ..., z_s\}$ are essential singularities

8 Analytic Continuation

Definition 58 (Analytic continuation) *Let $f : U \to \mathbb{C}$ analytic. An analytic continuation of f is an analytic function $g : V \to \mathbb{C}$ such that $U \cap V \neq 0$ and $g|_{U\cap V} = f|_{U\cap V}$*

Definition 59 *Let $f : U \to \mathbb{C}$ analytic, $w \in \mathbb{C}$, $z_0 \in U$, γ arc from z_0 to w. An analytic continuation of f to w along γ is an analytic function $g : V \to \mathbb{C}$ such that $\gamma \subset V$ and $g|_{U\cap V} = f|_{U\cap V}$*

Remark: In general the analytic continuation of f to w depends on the chosen arc γ

Theorem 50 *If f is analytic in a domain D_1 and g is a direct analytic continuation of f to the domain D_2, then the function*

$$F(z) = \begin{cases} f(z) & \text{for } z \text{ in } D_1 \\ g(z) & \text{for } z \text{ in } D_2 \end{cases}$$

is single-valued and analytic on $D_1 \cup D_2$

Theorem 51 (Monodromy Theorem) *Let $U \subset V$, $z_0 \in U$, $w \in V$, $f : U \to \mathbb{C}$ analytic, $\gamma_i \subset V$ arc from z_0 to w $(i = 0, 1)$ and $g_i; w_i \to \mathbb{C}$ analytic continuation of f to w along γ_i. If f can be extend along any arc $\gamma \subset V$ from z_0 to w and if γ_0 and γ_1 arc homotopic in V, then $g_0(w) = g_1(w)$.*

Definition 60 *An algebraic function is an analytic function $f : U \to \mathbb{C}$ such that there is a non-zero polynomial*

$$P(T_1, T_2) = \sum_{i,j=0}^{n} a_{i,j} T_1^i T_2^j$$

with $P(f(z), z) = 0$ for all $z \in U$

Theorem 52 *Let $f : U \to \mathbb{C}$ algebraic. then there are $w_1, ..., w_n \in \mathbb{C}$ such that f has an analytic continuation along every simple arc $\gamma \in \mathbb{C}\backslash\{w_1, ..., w_n\}$ that starts in U*

9 Conformal maps and the Rieman mapping theorem

Definition 61 *$f : U \to \mathbb{C}$ is n-to-1 if for all $w \in im(f)$, $\# f^{-1}(w) = n$*

Theorem 53 *$f : U \to \mathbb{C}$ analytic, non-constant $0 \in U$ and $\operatorname{ord}_0 f \geq 1$. Then $f|_{D^\circ_\epsilon(0)} : D^\circ_\epsilon(0) \to \mathbb{C}$ is n-to-1 for some $\epsilon > 0$*

Corollary 62 *$f : U \to \mathbb{C}$ analytic, non-constant $z_0 \in U$, $n = \operatorname{ord}_{z_0}(f - f(z_0)) \geq 1$. Then $f|_{D^\circ_\epsilon(0)} : D^\circ_\epsilon(0) \to \mathbb{C}$ is n-to-1 for some $\epsilon > 0$*

Definition 63 *A conformal map is an angle preserving map $f : U \to \mathbb{C}$*

Definition 64 *An analytic function $f : U \to \mathbb{C}$ is **conformal** if for every $w \in U$ there is an $a_w = r_w \cdot e^{it_w} \neq 0$ such that for every differentiable parametrized arc $z : [-1, 1] \to U$ with $z(0) = w$, $(f \circ z)'(0) = a_w \cdot z'(0)$*

Theorem 54 *$f : U \to \mathbb{C}$ analytic. If $f'(w) \neq 0$ for all $w \in U$, then f is conformal*

Corollary 65 *$f : U \to \mathbb{C}$ conformal, and $z_0 \in U$. Then $f|_{D^\circ_\epsilon(0)} : D^\circ_\epsilon(0) \to \mathbb{C}$ is injective for some $\epsilon > 0$*

Proposition 17 *$f : U \to V$ conformal, $v : V \to \mathbb{R}$ harmonic. Then $v \circ f : U \to V \to \mathbb{R}$*

Theorem 55 *$U, V \subset \mathbb{C}$ open, $f : U \to V$ analytic bujection and $g : V \to U$ inverse bihjection. Then f and g are conformal and $f'(z) \cdot g'(w) = 1$ for all $z \in U$ and $w = f(z)$*